ADDENDUM TO: “CONSTRUCTING QUANTIZED ENVELOPING ALGEBRAS VIA INVERSE LIMITS OF FINITE DIMENSIONAL ALGEBRAS”

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Abstract. It is shown that the question raised in Section 5.7 of [1] has an affirmative answer.

We use the notation and numbering from [1]. In Section 5.7 we raised the question: does ${}_R\mathbf{U}$ embed in ${}_R\widehat{\mathbf{U}}$? The purpose of this addendum is to show that the answer is affirmative. To be precise, we have the following result.

Theorem. *The map ${}_R\theta\colon {}_R\mathbf{U} \to {}_R\widehat{\mathbf{U}}$ defined in 5.7(a) is injective. Hence, ${}_R\mathbf{U}$ is isomorphic to the R-subalgebra of ${}_R\widehat{\mathbf{U}}$ generated by all $\widehat{E}_{\pm i}^{(m)}$ ($i \in I$, $m \geqslant 0$) and $\widehat{K}_h$ ($h \in Y$).*

Proof. Consider the commutative diagram of R-algebra maps

$$\begin{tikzcd}
{}_R\widehat{\mathbf{U}} \arrow[dr, "1\otimes\widehat{p}_{\pi'}"] \arrow[ddr, bend right, "1\otimes\widehat{p}_\pi"'] & & {}_R\mathbf{U} \arrow[ll, dotted, "{}_R\theta"'] \arrow[dl, "1\otimes p_{\pi'}"'] \arrow[ddl, bend left, "1\otimes p_\pi"] \\
& {}_R\mathbf{S}(\pi') \arrow[d, "1\otimes f_{\pi,\pi'}"] & \\
& {}_R\mathbf{S}(\pi) &
\end{tikzcd}$$

for any finite saturated $\pi \subset \pi'$. The universal property of inverse limits guarantees the existence of a unique R-algebra map ${}_R\theta\colon {}_R\mathbf{U} \to {}_R\widehat{\mathbf{U}}$ making the diagram commute, and one easily checks that this map coincides with the map defined in 5.7(a). We need to show that ${}_R\theta$ is injective.

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We note that from the definitions it follows that for any π the maps p_π and $\dot{p}_\pi$ are related by the identity $1_\lambda\, p_\pi(u)\, 1_\mu = \dot{p}_\pi(1_\lambda\, u\, 1_\mu)$, for any $u \in \mathbf{U}$, $\lambda, \mu \in X$. It follows immediately that

$$1_\lambda\,(1 \otimes p_\pi)(u)\, 1_\mu = (1 \otimes \dot{p}_\pi)(1_\lambda\, u\, 1_\mu),$$

for any $u \in {}_R\mathbf{U}$, $\lambda, \mu \in X$. This is needed below.

Let $u \in \ker {}_R\theta$ and $\lambda, \mu \in X$. Then $\widehat{1}_\lambda\, {}_R\theta(u)\, \widehat{1}_\mu = 0$ in ${}_R\widehat{\mathbf{U}}$. This implies that $1_\lambda\,(1 \otimes p_\pi)(u)\, 1_\mu = 0$ in ${}_R\mathbf{S}(\pi)$ for any π, and hence that $(1 \otimes \dot{p}_\pi)(1_\lambda\, u\, 1_\mu) = 0$ in ${}_R\mathbf{S}(\pi)$ for any π. Thus by Lemma 5.2 we have $1_\lambda\, u\, 1_\mu \in \cap_\pi\, {}_R\dot{\mathbf{U}}[\pi^c]$. Since the intersection is zero, the equality $1_\lambda\, u\, 1_\mu = 0$ holds in ${}_R\dot{\mathbf{U}}$, for any $\lambda, \mu \in X$. We claim this implies that $u = 0$.

To see the claim we observe that the construction of $\dot{\mathbf{U}}$ given in Section 3.1 and [3, Chapter 23] commutes with change of scalars. This is easily verified and left to the reader. It means that ${}_R\pi_{\lambda,\mu}(u) = 0$ where

$${}_R\pi_{\lambda,\mu} \colon {}_R\mathbf{U} \to {}_R\mathbf{U}/\Big(\sum_{h\in Y}(K_h - \xi^{\langle h,\lambda\rangle})\,{}_R\mathbf{U} + \sum_{h\in Y}{}_R\mathbf{U}(K_h - \xi^{\langle h,\mu\rangle})\Big)$$

is the canonical projection map. Thus it follows that

$$u \in \sum_{h\in Y}(K_h - \xi^{\langle h,\lambda\rangle})\,{}_R\mathbf{U} + \sum_{h\in Y}{}_R\mathbf{U}(K_h - \xi^{\langle h,\mu\rangle}).$$

Since this is true for all $\lambda, \mu \in X$ it follows that $u = 0$ as claimed. □

From [3, 31.1.5] we recall the category ${}_R\mathcal{C}$ of unital ${}_R\dot{\mathbf{U}}$-modules. As in [3, 23.1.4] one easily checks that this is the same as the category of ${}_R\mathbf{U}$-modules admitting a weight space decomposition. Following [3, 31.2.4], we say that an object M of ${}_R\mathcal{C}$ is *integrable* if for any $m \in M$ there exists some n_0 such that

$$E_i^{(n)}\, m = 0 = E_{-i}^{(n)}\, m$$

for all $n \geqslant n_0$.

We have the following consequence of the theorem, which generalizes [2, Proposition 5.11] and [3, Proposition 3.5.4].

Corollary. *Suppose that $u \in {}_R\mathbf{U}$ acts as zero on all integrable objects of ${}_R\mathcal{C}$. Then $u = 0$.*

Proof. The natural quotient map $1 \otimes p_\pi \colon {}_R\mathbf{U} \to {}_R\mathbf{S}(\pi)$ makes ${}_R\mathbf{S}(\pi)$ into a left ${}_R\mathbf{U}$-module, by defining $u \cdot s = \overline{u}\, s$ (for $u \in {}_R\mathbf{U}$, $s \in {}_R\mathbf{S}(\pi)$) where $\overline{u}$ is the image of u. It is easily checked that, as a left ${}_R\mathbf{U}$-module, ${}_R\mathbf{S}(\pi)$ is an integrable object of ${}_R\mathcal{C}$. Hence by hypothesis u acts as zero on ${}_R\mathbf{S}(\pi)$, for any finite saturated subset π of X^+. It follows

that u lies in the intersection of the kernels of the various $1 \otimes p_\pi$. By the commutative diagram above this implies that ${}_R\theta(u) = 0$. By the theorem, $u = 0$. $\square$

References

[1] S. Doty, Constructing quantized enveloping algebras via inverse limits of finite dimensional algebras, *J. Algebra* 321 (2009), 1225–1238.
[2] J.C. Jantzen, *Lectures on Quantum Groups*, Amer. Math. Soc. 1996.
[3] G. Lusztig, *Introduction to Quantum Groups*, Birkhäuser Boston 1993.

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